PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON TUESDAY, MARCH 5, 2024 on Sec 9.1: An Introduction to Differential Equations, INITIAL VALUE PROBLEMS (IVPs), and SEC. 9.3: SEPERABLE D.E.s

CLASS # 15

AN INTRODUCTION TO DIFFERENTIAL EQUATIONS (Ch, 9)

Definition: A Differential EQUATION (D.E.) is an equation that contains an unknown function and one or more of its derivatives.

For example, $y = y(t)$ represents an unknown function such that

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 2y = t + \cos(t).$$

This is a second-order D.E.

Function	EQUATION	Type
For $y = f(x)$,	$y'' - y' = 0$	2nd-order ordinary D.E. (ODE)
For $z = f(x,y)$,	$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$	Second-order Partial D.E.

The classification of DE's

$y''' + 4y = 2$ is a 3rd-order Ordinary D.E.

$(y')^2 - 3y = e^x$ is a 1st order O.DE.

A solution to a Differential Equation is a known function $y = f(x)$ such that, when that function and its derivatives are substituted for the unknown function and its derivatives, the EQUATION BECOMES A True Statement.

Ex: Consider the Ordinary DE

$$y'' + 4y = 0 \leftarrow \text{The zero function}$$

Problem Show that $y = 7\sin(2x)$ is a solution of this D.E.

Sol'n: $4y = 4(7\sin(2x)) = 28\sin(2x) = 4y$

$y' = 7(\cos(2x))2 = 14\cos(2x) = y'$

$y'' = 28(-\sin(2x)) = -28\sin(2x)$

$y'' + 4y = -28\sin(2x) + 28\sin(2x) = 0$

$\therefore y = 7\sin(2x)$ is a particular solution of the D.E.

A General Solution of a Differential Eqn is a parameterized family of functions, all of which are solutions and describes all the solutions of the D.E (except for a few singular solutions)

EXAMPLE: For $y'' + 4y = 0$,

The General Solution of this DE. is

$$y = C_1 \cos(2x) + C_2 \sin(2x)$$ where

C_1 and C_2 are real number constants.

Using $C_1 = 0$ and $C_2 = 7$ gives us

$$y = 7 \sin(2x).$$

FINDING the general solution for a Differential Equation that is in the Class of Differential called Separable D.E.

Separable (First-Order) Differential EQ's

$$\frac{dy}{dx} = g(x)\, f(y)$$

Ex: $y' = (3x^2+1)(y^2+3)$

$$y' = \frac{2x+1}{y+1} = (2x+1)\left(\frac{1}{y+1}\right)$$

$y' = \dfrac{1}{\cos(xy)}$ is not a separable D.E.

How to find the General Solution for a Separable D.E.

For $y' = g(x)\, f(y)$,

Rewrite it as $\frac{dy}{dx} = g(x)\, f(y)$

$$g(x)\,dx = \frac{1}{f(y)}\,dy = h(y)\,dy$$

$$h(y)\,dy = g(x)\,dx$$

$$\int h(y)\,dy = \int g(x)\,dx$$

$$H(y) = G(x) + C$$

and then we solve for y, if possible.

Ex: Solve the D.E. $\dfrac{dy}{dx} = \underbrace{-6x}\, \underbrace{y}$. $\left(\frac{1}{y}dy = -6x\,dx\right)$

Sol'n: First determine if "$y=0$" is a solution of the D.E.

For $y=0$, $\frac{dy}{dx} = 0$ and $(-6x)y = (-6x)(0) = 0$

$\hookrightarrow$ $\frac{dy}{dx} = 0$ and $-6xy = 0$, "$y=0$" is a solution to the D.E.

Assume $y \neq 0$. $\dfrac{dy}{dx} = -6xy$

$$\frac{1}{y}\,dy = -6x\,dx$$

$\int \frac{1}{y}\,dy = \int(-6x)\,dx = -3x^2 + C$ where C is any real number

$\ln|y| = -3x^2 + C$

$e^{\ln|y|} = e^{(-3x^2+C)} = e^{-3x^2}\cdot e^{C}$, Write $c_1 = e^{c}$

$|y| = c_1 e^{-3x^2}$ where $c_1 > 0$, since $c_1 = e^{c}$.

$y = \pm c_1 e^{-3x^2} = A e^{-3x^2}$, where $A \neq 0$.

this = that $\Rightarrow$ $e^{\text{this}} = e^{\text{that}}$

$|x| = 5 \Rightarrow x = \pm 5$

Since $y=0$ is also a solution, we allow $A=0$.

$\boxed{\text{The General Solution of the D.E } \frac{dy}{dx} = -6xy \text{ is } y = Ae^{-3x^2} \text{ where } A \text{ is any real number constant}}$

An Initial VALUE PROBLEM, (IVP),
consists of a D.E. and particular values
that the solution of interest must have.
These values are called "Initial Conditions":

Example of an IVP: Find a solution of the D.E.

$$y'' + 4y = 0 \quad \text{such that}$$
$$y(\pi/4) = 3 \quad \text{and} \quad y'(\pi/4) = 10.$$

Sol'n: Recall the general solution of this D.E is

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$
$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x).$$

Set $x = \pi/4$ and solve for c_1 and c_2:

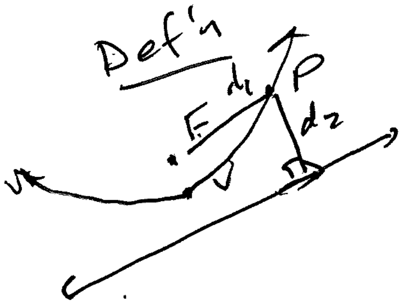
$$3 = y(\pi/4) = c_1 \cos(\pi/2) + c_2 \sin(\pi/2)$$
$$= c_1 * 0 + c_2 * 1 = c_2 = 3$$

$$10 = y'(\pi/4) = -2c_1 \sin(\pi/2) + 2c_2 \cos(\pi/2)$$
$$= -2c_1 * 1 + 0 = -2c_1 = 10$$
$$c_1 = -5$$

The solution of this IVP is
$$\boxed{y = -5 * \cos(2x) + 3 \sin 2x}$$

The Reflective Properties of Conics

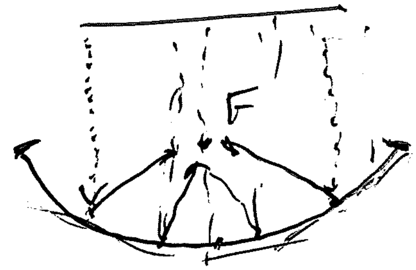
Def'n



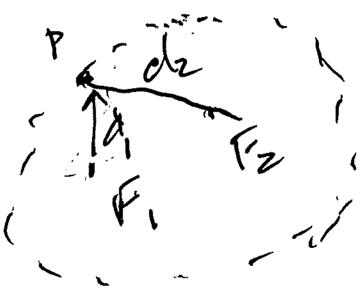
For Parabolas

$d_1 = d_2$

Reflection Property

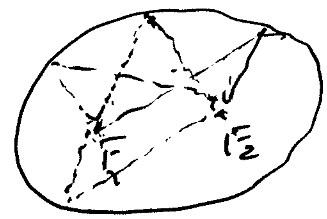


For Ellipses

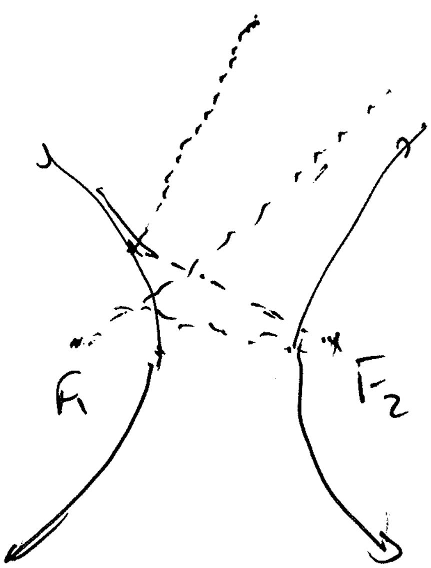


$L = 2a$

$d_1 + d_2 = L$



For Hyperbolas



$L = 2a$

$|d_1 - d_2| = L$